2015年 理系 第5問

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$\boxed{5}$ nは自然数，aは$a>\dfrac{3}{2}$をみたす実数とし，実数xの関数

$$f(x)=\int_0^x (x-\theta)(a\sin^{n+1}\theta-\sin^{n-1}\theta)\,d\theta$$

を考える．ただし，$n=1$のときは$\sin^{n-1}\theta=1$とする．

(1) $\displaystyle\int_0^{\frac{\pi}{2}}\sin^{n+1}\theta\,d\theta=\frac{n}{n+1}\int_0^{\frac{\pi}{2}}\sin^{n-1}\theta\,d\theta$を示せ．

(2) $f'\left(\dfrac{\pi}{2}\right)=0$をみたす$n$と$a$の値を求めよ．

(3) (2)で求めたnとaに対して，$f\left(\dfrac{\pi}{2}\right)$を求めよ．

(1) $I=\displaystyle\int_0^{\frac{\pi}{2}}\sin^{n+1}\theta\,d\theta$ とおくと，

$$I=\int_0^{\frac{\pi}{2}}(-\cos\theta)'\sin^n\theta\,d\theta$$

$$=\Big[-\cos\theta\sin^n\theta\Big]_0^{\frac{\pi}{2}}+\int_0^{\frac{\pi}{2}}\cos\theta\cdot n\sin^{n-1}\theta\cdot\cos\theta\,d\theta$$

$$=\int_0^{\frac{\pi}{2}}n(\sin^{n-1}\theta-\sin^{n+1}\theta)\,d\theta$$

$$=n\int_0^{\frac{\pi}{2}}\sin^{n-1}\theta\,d\theta-nI$$

$$\therefore (n+1)I=n\int_0^{\frac{\pi}{2}}\sin^{n-1}\theta\,d\theta$$

$$\therefore \int_0^{\frac{\pi}{2}}\sin^{n+1}\theta\,d\theta=\frac{n}{n+1}\int_0^{\frac{\pi}{2}}\sin^{n-1}\theta\,d\theta \quad ▨$$

(2) $$f(x)=x\int_0^x a\sin^{n+1}\theta-\sin^{n-1}\theta\,d\theta-\int_0^x\theta(a\sin^{n+1}\theta-\sin^{n-1}\theta)\,d\theta$$

$$\therefore f'(x)=\int_0^x a\sin^{n+1}\theta-\sin^{n-1}\theta\,d\theta+x(a\sin^{n+1}x-\sin^{n-1}x)-x(a\sin^{n+1}x-\sin^{n-1}x)$$

$$=\int_0^x a\sin^{n+1}\theta-\sin^{n-1}\theta\,d\theta$$

$\therefore$ (1)より，$f'(\frac{\pi}{2})=a\cdot\dfrac{n}{n+1}\displaystyle\int_0^{\frac{\pi}{2}}\sin^{n-1}\theta\,d\theta-\int_0^{\frac{\pi}{2}}\sin^{n-1}\theta\,d\theta$

$$=\frac{an-n-1}{n+1}\int_0^{\frac{\pi}{2}}\sin^{n-1}\theta\,d\theta$$

$0<\theta<\frac{\pi}{2}$ において，$\sin^{n-1}\theta>0$より $\displaystyle\int_0^{\frac{\pi}{2}}\sin^{n-1}\theta\,d\theta>0$

$\therefore f'(\frac{\pi}{2})=0$ となるのは，$an-n-1=0$ のとき $\therefore a=1+\frac{1}{n}$

$a>\frac{3}{2}$ かつ n：自然数より，

$\frac{1}{n}>\frac{1}{2}$ $\therefore \underline{n=1}$ //

このとき $\underline{a=2}$ //

(3) (2)より，

$$f'(x)=\int_0^x 2\sin^2\theta-1\,d\theta$$

$$=\int_0^x 2\cdot\frac{1-\cos2\theta}{2}-1\,d\theta$$

$$=\Big[-\frac{1}{2}\sin2\theta\Big]_0^x$$

$$=-\frac{1}{2}\sin2x$$

$\therefore f(x)=\frac{1}{4}\cos2x+C$ （Cは積分定数）

と表せる $f(0)=0$より $C=-\frac{1}{4}$

$\therefore f(x)=\frac{1}{4}\cos2x-\frac{1}{4}$

$\therefore \underline{f(\frac{\pi}{2})=-\frac{1}{2}}$ //